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MA111 - Engineering Mathematics - II
Problem Sheet - 2

Infinite Series, Integral and Comparison Tests

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1. Is $\sum_{n=1}^{\infty} \frac{7^{n+1}}{9^n}$ is convergent? (Hint: Observe that the n-th term is $7\left(\frac{7}{9}\right)^n$)

2. Use the knowledge of infinite series to conclude that $\frac{n}{2^n} \to 0$.

3. Show that the series $\sum_{n=1}^{\infty} \frac{1}{2^n - n}$ is convergent. (Hint: Observe that $2^n - n \geq 2^n - 2^{n-1} = 2^{n-1}$)

4. Find a formula for the n-th partial sum of the series and use it to determine if the series converges or diverges. If a series converges, find its sum.

   (a) $\sum_{n=1}^{\infty} \left(\frac{3}{n^2} - \frac{3}{(n+1)^2}\right)$

   (b) $\sum_{n=1}^{\infty} (\tan(n) - \tan(n-1))$

   (c) $\sum_{n=1}^{\infty} \left(\frac{6}{(2n-1)(2n+1)}\right)$

   (d) $\sum_{n=1}^{\infty} \left(\frac{2n+1}{n^2(n+1)^2}\right)$

   (e) $\sum_{n=1}^{\infty} \left(\sqrt{n+4} - \sqrt{n+3}\right)$

   (a) converges to 3 (b) diverges (c) converges to 3 (d) converges to 1 (e) diverges

5. Find convergent geometric series $A = \sum a_n$ and $B = \sum b_n$ that illustrate the fact that $\sum a_n b_n$ may converge without being equal to AB.

6. If $\sum a_n$ converges and $a_n > 0$ for all n, can anything be said about $\sum \left(\frac{1}{a_n}\right)$?

7. If $\sum a_n$ converges and $\sum b_n$ diverges, can anything be said about their term-by-term sum $\sum(a_n + b_n)$?

8. Find the value of a for which
$$1 + e^a + e^{2a} + e^{3a} + \cdots = 5.$$
   (Ans: $a = \ln\left(\frac{4}{5}\right)$)

9. Discuss the converges of the following series using of the integral test

   (a) The series $\sum_{n=1}^{\infty} \frac{1}{(n+2)\ln(n+2)}$ diverges. (Hint: $f(x) = \frac{1}{x \ln x}$)

(b) Show that the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ is convergent if $p > 0$. (Hint: $f(x) = \frac{\ln x}{x^p}$)

(c) Discuss the convergence of the series $\sum_{n=1}^{\infty} n\, e^{-n^2}$ (Hint: $f(x) = x\, e^{-x^2}$)

(d) $\sum_{n=2}^{\infty} \frac{n-4}{n^2-2n+1}$ (Hint: $f(x) = \frac{x-4}{x^2-2x+1}$)

(e) $\sum_{n=2}^{\infty} \frac{1}{5n+10\sqrt{n}}$

10. For what values of a, if any, do the following series converge?

(a) $\sum_{n=1}^{\infty} \left(\frac{a}{n+2} - \frac{1}{n+4} \right)$

(b) $\sum_{n=3}^{\infty} \left(\frac{1}{n-1} - \frac{2a}{n+1} \right)$
(Hint: Apply integral test)

11. **The Cauchy condensation test**: Let $\{a_n\}$ be a nonincreasing sequence ($a_n \geq a_{n+1}$ for all n) of positive terms that converges to 0. Then $\sum a_n$ converges if and only if $\sum 2^n a_{2^n}$ converges. Use Cauchy condensation test to show that

(a) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges

(b) $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is converges if $p > 1$ and diverges if $p \leq 1$

12. Discuss the convergence of the series using comparison test

(a) $\sum_{n=1}^{\infty} \frac{\sqrt{(n+1)} - \sqrt{n}}{n^p}$

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{(n^3+1)}}$

(c) $\sum_{n=1}^{\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^4+1}$

(d) $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \cdots$

(e) Show that $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$ (Hint: Use the convergence of $\sum \frac{1}{n^2}$)

13. Suppose that $a_n > 0$ and $\lim_{n\to\infty} n^2 a_n = 0$. Prove that $\sum a_n$ converges.

14. $\sum a_n$ is a convergent series of non-negative numbers, can anything be said about $\sum_{n=1}^{\infty} \left(\frac{a_n}{n} \right)$?

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